ON THE EQUATION $P(f) = Q(g)$, WHERE P, Q ARE POLYNOMIALS AND f, g ARE ENTIRE FUNCTIONS

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ABSTRACT. In 1922 Ritt described polynomial solutions of the functional equation $P(f) = Q(g)$. In this paper we describe solutions of the equation above in the case when P, Q are polynomials while f, g are allowed to be arbitrary entire functions. In fact, we describe solutions of the more general functional equation $s = P(f) = Q(g)$, where s, f, g are entire functions and P, Q are arbitrary rational functions. As an application we solve the problem of description of "strong uniqueness polynomials" for entire functions.

1. INTRODUCTION

In this paper we describe all possible solutions of the functional equation

$$P \circ f = Q \circ g, \tag{1}$$

where P, Q are polynomials, f, g are entire functions, and the symbol $\circ$ denotes the superposition of functions, $f_1 \circ f_2 = f_1(f_2)$. In fact we describe solutions of the more general functional equation

$$s = P \circ f = Q \circ g, \tag{2}$$

where s, f, g are entire functions and P, Q are arbitrary rational functions.

Let us give several examples of solutions of (1). First of all observe that for any polynomial P and any entire function f one can obtain a solution of (1) setting

$$Q = P \circ \alpha, \quad g = \alpha^{-1} \circ f, \tag{3}$$

where α is a linear Möbius transformation. Observe also that if P, Q, f, g is a solution of (1), then for any entire function h and any polynomial U the collection

$$\hat{P} = U \circ P, \quad \hat{Q} = U \circ Q, \quad \hat{f} = f \circ h, \quad \hat{g} = g \circ h$$

also is a solution of (1).

In order to lighten the notation, in case if rational functions P, Q and entire functions s, f, g such that (2) holds satisfy (3) for some Möbius transformation α, we will say that that the decomposition $P \circ f$ of s is equivalent to the decomposition $Q \circ g$. For equivalent decompositions we will use the notation $P \circ f \sim Q \circ g$.

The simplest examples of solutions of (1) for which the decompositions $P \circ f$ and $Q \circ g$ are not equivalent are provided by polynomials. For example, we have $z^n \circ z^m = z^m \circ z^n$. More generally, for any polynomial R and $r \geq 0$, $n \geq 1$ the equality

$$z^n \circ z^r R(z^n) = z^r R^n(z) \circ z^n \tag{4}$$

holds. Another examples of polynomial solutions of (1) are provided by the Chebyshev polynomials T_n defined by the equality

$$(5) \qquad\qquad\qquad T_n(\cos z) = \cos nz.$$

Indeed, it follows from (5) that for any $m, n \geq 1$ we have:

$$(6) \qquad\qquad\qquad T_n \circ T_m = T_m \circ T_n.$$

The theory of functional decompositions of polynomials developed by Ritt [27] yields that actually any polynomial solution of (1) in a sense reduces either to (4) or to (6). Namely, the following statement is true: if polynomials P, Q, f, g satisfy (1) then there exist polynomials U, $\tilde{P}$, $\tilde{Q}$, $\tilde{f}$, $\tilde{g}$, h such that

$$(7) \quad P = U \circ \tilde{P}, \quad Q = U \circ \tilde{Q}, \quad f = \tilde{f} \circ h, \quad g = \tilde{g} \circ h, \quad \tilde{P} \circ \tilde{f} = \tilde{Q} \circ \tilde{g}$$

and up to a possible replacement of $\tilde{P}$ by $\tilde{Q}$ and $\tilde{f}$ by $\tilde{g}$ either

$$(8) \qquad\qquad \tilde{P} \circ \tilde{f} \sim z^n \circ z^r R(z^n), \qquad \tilde{Q} \circ \tilde{g} \sim z^r R^n(z) \circ z^n,$$

where R is a polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$, or

$$(9) \qquad\qquad \tilde{P} \circ \tilde{f} \sim T_n \circ T_m, \qquad \tilde{Q} \circ \tilde{g} \sim T_m \circ T_n,$$

where T_n, T_m are the corresponding Chebyshev polynomials with $n, m \geq 1$, and $\mathrm{GCD}(n, m) = 1$.

The simplest example of a solution of (1) with transcendental f, g is provided by the equality

$$\cos^2 z = 1 - \sin^2 z.$$

More generally, for any polynomial S we have:

$$(10) \qquad\qquad z^2 \circ \cos z\, S(\sin z) = (1 - z^2) S^2(z) \circ \sin z.$$

The equality

$$(11) \qquad\qquad\qquad T_n \circ \cos mz = T_m \circ \cos nz$$

also is an example of a solution of (1). Nevertheless, in a sense this equality is a corollary of equality (6) since

$$\cos mz = T_m \circ \cos z, \qquad \cos nz = T_n \circ \cos z.$$

On the other hand, for example the equality

$$-T_2 \circ \cos\left(\frac{\pi}{2} + z\right) = T_2 \circ \cos z$$

already can not be reduced in a similar way to (4), (6), or (10). More generally, for any $m, n \geq 1$, $l > 1$, and $0 \leq k < nl$ we have:

$$(12) \qquad\qquad -T_{nl} \circ \cos\left(\frac{(2k+1)\pi}{nl} + mz\right) = T_{ml} \circ \cos(nz).$$

Our first result states that up to one "sporadic" exception any solution of (1) can be reduced to (4), (6), (10) or (12).

Theorem A. *Suppose that polynomials P, Q and entire functions f, g satisfy the equation*

$$P \circ f = Q \circ g.$$

Then there exist polynomials F, $\tilde{P}$, $\tilde{Q}$ and entire functions $\tilde{f}$, $\tilde{g}$, t such that

$$P = F \circ \tilde{P}, \quad Q = F \circ \tilde{Q}, \quad f = \tilde{f} \circ t, \quad g = \tilde{g} \circ t, \quad \tilde{P} \circ \tilde{f} = \tilde{Q} \circ \tilde{g}$$

and, up to a possible replacement of P by Q and f by g, one of the following conditions holds:

1)
$$\tilde{P} \circ \tilde{f} \sim z^n \circ z^r R(z^n), \qquad \tilde{Q} \circ \tilde{g} \sim z^r R^n(z) \circ z^n,$$

where R is a polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$;

2)
$$\tilde{P} \circ \tilde{f} \sim T_n \circ T_m, \qquad \tilde{Q} \circ \tilde{g} \sim T_m \circ T_n,$$

where T_n, T_m are the corresponding Chebyshev polynomials with $m, n \geq 1$, and $\mathrm{GCD}(n, m) = 1$;

3)
$$\tilde{P} \circ \tilde{f} \sim z^2 \circ \cos z \, S(\sin z), \qquad \tilde{Q} \circ \tilde{g} \sim (1 - z^2) \, S^2(z) \circ \sin z,$$

where S is a polynomial;

4)
$$\tilde{P} \circ \tilde{f} \sim -T_{nl} \circ \cos\left(\frac{(2k+1)\pi}{nl} + mz\right), \qquad \tilde{Q} \circ \tilde{g} \sim T_{ml} \circ \cos(nz),$$

where T_{nl}, T_{ml} are the corresponding Chebyshev polynomials with $m, n \geq 1$, $\mathrm{GCD}(n, m) = 1$, $l > 1$, and $0 \leq k < nl$;

5)
$$\tilde{P} \circ \tilde{f} \sim (z^2 - 1)^3 \circ \left(\frac{i \sin 2x + 2\sqrt{2}\cos x}{\sqrt{3}}\right),$$

$$\tilde{Q} \circ \tilde{g} \sim (3z^4 - 4z^3) \circ \left(\frac{i \sin 3x}{3\sqrt{2}} + \cos 2x + \frac{i \sin x}{\sqrt{2}} + \frac{2}{3}\right).$$

Since a composition $P \circ f$ of a polynomial P and an entire function f is an entire function, the problem of description of solutions of equation (1) is a particular case of the problem of description of all possible "double decompositions" (2) of an entire function. Notice that different aspects of the theory of decompositions of entire functions were studied in many recent papers (see e. g. [8], [17], [19], [20], [21], [22]). However, this theory is still far from its completion. In particular, there exist no results about double decompositions of entire functions similar to the results of Ritt.

Our next result describes solutions of equation (2) in case where the functions P, Q are rational and at least one of them is not a polynomial. Together with Theorem A this provides a complete description of solutions of equation (2) with rational P, Q and entire s, f, g.

Theorem B. *Suppose that rational functions P, Q and entire functions s, f, g satisfy the equation*

$$s = P \circ f = Q \circ g.$$

Furthermore, suppose that at least one of the functions P, Q is not a polynomial. Then there exist rational functions F, $\tilde{P}$, $\tilde{Q}$ and entire functions $\tilde{f}$, $\tilde{g}$, t such that

$$P = F \circ \tilde{P}, \quad Q = F \circ \tilde{Q}, \quad f = \tilde{f} \circ t, \quad g = \tilde{g} \circ t, \quad \tilde{P} \circ \tilde{f} = \tilde{Q} \circ \tilde{g}$$

and, up to a possible replacement of P by Q and f by g, one of the following conditions holds:

1)
$$\tilde{P} \circ \tilde{f} \sim z^n \circ e^{rz} L(e^{nz}), \qquad \tilde{Q} \circ \tilde{g} \sim z^r L^n(z) \circ e^{nz},$$

where L is a Laurent polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$;

2)
$$\tilde{P} \circ \tilde{f} \sim T_n \circ \cos(mz), \qquad \tilde{Q} \circ \tilde{g} \sim \frac{1}{2}\left(z^m + \frac{1}{z^m}\right) \circ e^{inz},$$

where T_n is the Chebyshev polynomial, $m, n \geq 1$, and $\mathrm{GCD}(n, m) = 1$.

Yet another problem related to equation (1) is the problem of description of "strong uniqueness polynomials" for entire functions. Recall that a polynomial P is called a strong uniqueness polynomial for entire functions if the equality

(13)
$$P \circ f = cP \circ g,$$

where f, g are entire functions and $c \in \mathbb{C}$, implies that $c = 1$ and $f \equiv g$. Such polynomials are closely related to the "uniqueness range sets" for entire functions and were studied in the recent papers [6], [12], [14], [28], [30] (see also the papers [1], [2], [13], [15], [16], [24] where the similar question was studied for meromorphic functions).

Usually, the problem of description of strong uniqueness polynomials for entire functions is studied under additional conditions of "generic position" imposed on the polynomial P. Our last result based on Theorem A provides a complete solution of this problem in the general case.

Theorem C. *A polynomial P is not a strong uniqueness polynomial for entire functions if and only if there exists a Möbius transformation α such that either*

1)
$$P = z^r R(z^n) \circ \alpha,$$

where R is a polynomial and $r \geq 0$, $n > 1$, or

2)
$$P = F \circ T_l \circ \alpha,$$

where F is a polynomial, T_l is the Chebyshev polynomial, and $l > 1$.

The paper is organized as follows. In the second section, using a result about parametrizations of algebraic curves by entire functions obtained in [4], [11], [22] we relate the classification of solutions of equation (2) with the classification of double decompositions of Laurent polynomials into compositions of rational functions. In the third section we review the papers [5], [23], [25], [31] where such a classification was obtained. Finally, in the fourth section we prove Theorems A, B, and C.

2. REDUCTION

Let $\mathcal{E} : u(x, y) = 0$ be an affine algebraic curve. Recall, that a pair f, g of functions meromorphic on a simply connected domain D of $\mathbb{CP}^1$ is called a meromorphic parametrization of $\mathcal{E}$ on D if for any point $z \in D$ which is not a pole of f or g the equality $u(f(z), g(z)) = 0$ holds and with finitely many exceptions any point of $\mathcal{E}$ is of the form $(f(z), g(z))$ for some $z \in D$. If $D = \mathbb{C}$ and functions f, g are entire then the corresponding parametrization is called entire.

Denote by $\hat{\mathcal{E}}$ the desingularization of the curve $\mathcal{E}$. The general structure of meromorphic parametrizations of $\mathcal{E}$ on D is described by the following theorem (see [4]).

Theorem 2.1. *Let f, g be a meromorphic parametrization of $\mathcal{E}$ on D. Then there exist a holomorphic function $h : D \to \hat{\mathcal{E}}$, meromorphic functions $U, V : \hat{\mathcal{E}} \to \mathbb{CP}^1$, and a finite set $S \subset \hat{\mathcal{E}}$ such that $f = U \circ h$, $g = V \circ h$ and the mapping $(U, V) : \hat{\mathcal{E}} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is injective on $\hat{\mathcal{E}} \setminus S$.* □

The class of curves having a meromorphic parametrization on $\mathbb{C}$ is quite restrictive in view of the following classical Picard theorem [26].

Theorem 2.2. *If $\mathcal{E}$ has a meromorphic parametrization on $\mathbb{C}$ then $\hat{\mathcal{E}}$ has genus zero or one.* □

Furthermore, for curves having an entire parametrization the following much more precise result holds (see [11], [4]).

Theorem 2.3. *Let f, g be an entire parametrization of $\mathcal{E}$. Then $\hat{\mathcal{E}} = \mathbb{CP}^1$ and there exist an entire function h, rational functions U, V, and a finite set $S \subset \mathbb{CP}^1$ such that $f = U \circ h$, $g = V \circ h$ and the mapping $(U, V) : \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is injective on $\mathbb{CP}^1 \setminus S$.* □

Theorem 2.3 permits to relate the description of solutions of (2) with the description of solutions of the equation

$$(14) \qquad L = P \circ U = Q \circ V,$$

where L is a Laurent polynomial and P, Q, U, V are rational functions.

Theorem 2.4. *Suppose that rational functions P, Q and entire functions s, f, g satisfy equation (2). Then there exist an entire function h, a Laurent polynomial L, and rational functions U, V such that*

$$(15) \qquad s = L \circ h, \quad f = U \circ h, \quad g = V \circ h, \quad L = P \circ U = Q \circ V.$$

Proof. If (2) holds, then f, g is an entire parametrization of a factor of the algebraic curve

$$P_1(x)Q_2(y) - P_2(x)Q_1(y) = 0,$$

where P_1, P_2 and Q_1, Q_2 are pairs polynomials without common roots such that

$$P = P_1/P_2, \quad Q = Q_1/Q_2,$$

and hence by Theorem 2.3 there exist an entire function h and rational functions U, V such that

$$(16) \qquad f = U \circ h, \quad g = V \circ h.$$

Furthermore, it follows from $P \circ f = Q \circ g$ that $P \circ U = Q \circ V$. Besides, clearly $s = L \circ h$, where $L = P \circ U = Q \circ V$.

Since s is an entire function, the equality $s = L \circ h$ implies that h does not take any value in $\mathbb{C}$ which is a pole of L. On the other hand, by the Picard Little theorem an entire function may omit at most one value a in $\mathbb{C}$. Therefore, L is either a polynomial or has at most one pole in $\mathbb{C}$ and in the last case this pole necessarily coincides with a. Therefore, replacing if necessary h by $h - a$ and L by $L \circ (z + a)$, without loss of generality we may assume that L is a Laurent polynomial. $\square$

3. Decompositions of Laurent polynomials

In this section we review results concerning double decompositions of polynomials and Laurent polynomials. In accordance with the notation introduced above, if P, Q, U, V are rational functions such that (14) holds for some rational function L and

$$P = Q \circ \alpha, \quad U = \alpha^{-1} \circ V$$

for some Möbius transformation α, then we will say that the decomposition $P \circ U$ of L is equivalent to the decomposition $Q \circ V$ and will use for equivalent decompositions the notation $P \circ U \sim Q \circ V$.

The decomposition theory of polynomials was constructed by Ritt in the paper [27]. In particular, Ritt proved that if $\tilde{P}, \tilde{Q}, \tilde{f}, \tilde{g}$ are polynomials satisfying

$$\tilde{P} \circ \tilde{f} = \tilde{Q} \circ \tilde{g}$$

and such that

$$(17) \qquad \text{GCD}(\deg \tilde{P}, \deg \tilde{Q}) = 1, \quad \text{GCD}(\deg \tilde{f}, \deg \tilde{g}) = 1,$$

then up to a possible replacement of $\tilde{P}$ by $\tilde{Q}$ and $\tilde{f}$ by $\tilde{g}$ either (8) or (9) holds.

On the other hand, it was proved in [7], [29] (see also [18]) that if P, Q, f, g are arbitrary polynomials satisfying (1), then there exist polynomials $U, \tilde{P}, \tilde{Q}, \tilde{f}, \tilde{g}, h$ such that

$$(18) \qquad \deg U = \text{GCD}(\deg P, \deg Q), \quad \deg h = \text{GCD}(\deg f, \deg g)$$

and equalities (7) hold. Clearly, this result and the Ritt theorem taken together provide a full description of polynomial solutions of (1).

It is easy to see that the problem of description of polynomial solutions of equation (1) essentially is equivalent to the problem of description of algebraic curves of the form

$$(19) \qquad P(x) - Q(y) = 0$$

having a factor of genus zero with one point at infinity. Indeed, if (19) is such a curve, then the corresponding factor can be parametrized by some polynomials f, g implying (1), and vice versa if P, Q, f, g is a polynomial solution of (1), then (19) has a factor of genus zero with one point at infinity. A more general problem of description of curves (19) having a factor of genus 0 with at most two points at infinity is closely related to the number theory and was studied in the papers [10], [5]. In particular, in [5] an explicit list of such curves, defined over any field k of characteristic zero, was obtained.

It was observed by the author several years ago that the problem of description of solutions of equation (14), where L is a Laurent polynomial and P, Q, U, V are rational functions, mostly reduces to the above mentioned problem of description of

curves (19) having a factor of genus 0 with at most two points at infinity. Indeed, since a Laurent polynomial L has at most two poles, it is easy to see that any decomposition of L into a composition of two rational functions is equivalent either to a decomposition $A \circ L_1$, where A is a polynomial and L_1 is a Laurent polynomial, or to a decomposition $L_2 \circ B$, where L_2 is a Laurent polynomial and $B = cz^d$, $c \in \mathbb{C}$, $d \geq 1$. Therefore, a description of solutions of (14) reduces to a description of solutions of the following three equations:

$$(20) \qquad A \circ L_1 = B \circ L_2,$$

where A, B are polynomials and L_1, L_2 are Laurent polynomials,

$$(21) \qquad A \circ L_1 = L_2 \circ z^d,$$

where A is a polynomial and L_1, L_2 are Laurent polynomials, and

$$(22) \qquad L_1 \circ z^{d_1} = L_2 \circ z^{d_2},$$

where L_1, L_2 are Laurent polynomials.

Equation (22) is very simple. Equation (21) is more complicated but still can be analysed quite easily in view of the presence of symmetries. Finally, if A, B, L_1, L_2 is a solution of equation (20), then the curve

$$(23) \qquad A(x) - B(y) = 0$$

has a factor of genus 0 with at most two points at infinity and vice versa for any such a curve its factor of genus 0 may be parametrized by some Laurent polynomials.

A comprehensive self-contained theory of decompositions of Laurent polynomials was constructed in [25] where the equation

$$A \circ C = B \circ D$$

was studied in a more general setting involving holomorphic functions on compact Riemann surfaces. In particular, in [25] were proposed new proofs of the Ritt theorem and the classification of curves (19) having a factor of genus zero with at most two points at infinity. Another approach to the problem may be found in [31] where solutions of (20) and (21) are deduced from results of [5] and [3].

Below we collect necessary results from [25]. In order to lighten the notation set

$$U_n(z) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \quad V_n(z) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right).$$

We start from the description of solutions of equation (20).

Theorem 3.1. *Suppose that polynomials A, B and Laurent polynomials L_1, L_2 satisfy the equation*

$$A \circ L_1 = B \circ L_2.$$

Then there exist polynomials $E, \tilde{A}, \tilde{B}$ and Laurent polynomials $W, \tilde{L}_1, \tilde{L}_2$ such that

$$A = E \circ \tilde{A}, \quad B = E \circ \tilde{B}, \quad L_1 = \tilde{L}_1 \circ W, \quad L_2 = \tilde{L}_2 \circ W, \quad \tilde{A} \circ \tilde{L}_1 = \tilde{B} \circ \tilde{L}_2$$

and, up to a possible replacement of A by B and L_1 by L_2, one of the following conditions holds:

$$1) \qquad \tilde{A} \circ \tilde{L}_1 \sim z^n \circ z^r R(z^n), \qquad \tilde{B} \circ \tilde{L}_2 \sim z^r R^n(z) \circ z^n,$$

where R is a polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(n, r) = 1$;

2) $$\tilde{A} \circ \tilde{L}_1 \sim T_n \circ T_m, \qquad \tilde{B} \circ \tilde{L}_2 \sim T_m \circ T_n,$$

where T_n, T_m are the corresponding Chebyshev polynomials with $m, n \geq 1$, and $\mathrm{GCD}(n, m) = 1$;

3) $$\tilde{A} \circ \tilde{L}_1 \sim z^2 \circ U_1 S(V_1), \qquad \tilde{B} \circ \tilde{L}_2 \sim (1 - z^2) S^2 \circ V_1,$$

where S is a polynomial;

4) $$\tilde{A} \circ \tilde{L}_1 \sim -T_{nl} \circ U_m(\varepsilon z), \qquad \tilde{B} \circ \tilde{L}_2 \sim T_{ml} \circ U_n,$$

where T_{nl}, T_{ml} are the corresponding Chebyshev polynomials with $m, n \geq 1$, $l > 1$, $\varepsilon^{nlm} = -1$, and $\mathrm{GCD}(n, m) = 1$;

5) $$\tilde{A} \circ \tilde{L}_1 \sim (z^2 - 1)^3 \circ \left(\frac{i}{\sqrt{3}} V_2 + \frac{2\sqrt{2}}{\sqrt{3}} U_1 \right),$$

$$\tilde{B} \circ \tilde{L}_2 \sim (3z^4 - 4z^3) \circ \left(\frac{i}{3\sqrt{2}} V_3 + U_2 + \frac{i}{\sqrt{2}} V_1 + \frac{2}{3} \right). \quad \square$$

The expressions for A and B given in Theorem 3.1 coincide with the ones given in [5], Theorem 9.3 (for $k = \mathbb{C}$) and [25], Theorem 1.1 (see also Theorem 7.2 and Theorem 8.1 of [25]). The expressions for L_1, L_2 coincide with the ones given in [25] in 1), 2), 4) and slightly differ in 3) and 5). Say, in [25] in 3) we used for the curve

$$x^2 - (1 - y^2) S(y) = 0$$

the parametrization

$$L_1 = \frac{z^2 - 1}{z^2 + 1} S\left(\frac{2z}{z^2 + 1}\right), \qquad L_2 = \frac{2z}{z^2 + 1},$$

while now the parametrization

$$L_1 = U_1 S(V_1), \qquad L_2 = V_1.$$

A similar change is made in 5). Since the rational functions in the new parametrizations have the same degrees as the corresponding functions in the old parametrizations, the new parametrizations can be obtained from the old ones by composing them with Möbius transformations, and therefore such a change of parametrizations does not affect the conclusion of the theorem.

The solutions of equation (21) (in the case where this equation does not reduce to (20)) are described by the following theorem (see [25], Theorem 6.4).

Theorem 3.2. *Suppose that polynomials A, B and Laurent polynomials L_1, L_2 (which are not polynomials) satisfy the equation*

$$A \circ L_1 = L_2 \circ B.$$

Then there exist polynomials E, $\tilde{A}$, $\tilde{B}$, W and Laurent polynomials $\tilde{L}_1$, $\tilde{L}_2$ such that

$$A = E \circ \tilde{A}, \quad L_2 = E \circ \tilde{L}_2, \quad L_1 = \tilde{L}_1 \circ W, \quad B = \tilde{B} \circ W, \quad \tilde{A} \circ \tilde{L}_1 = \tilde{L}_2 \circ \tilde{B}$$

and either

(24) $$\tilde{A} \circ \tilde{L}_1 \sim z^n \circ z^r L(z^n), \qquad \tilde{L}_2 \circ \tilde{B} \sim z^r L^n(z) \circ z^n,$$

where L is a Laurent polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(r, n) = 1$, or

$$(25) \qquad \tilde{A} \circ \tilde{L}_1 \sim T_n \circ U_m, \qquad \tilde{L}_2 \circ \tilde{B} \sim U_m \circ z^n,$$

where T_n is the Chebyshev polynomial, $n \geq 1$, $m \geq 1$, and $\mathrm{GCD}(m, n) = 1$. $\qquad\square$

Finally, solutions of equation (22) are described as follows (see e.g. [25], Lemma 6.1).

Lemma 3.1. *Let L_1, L_2 be Laurent polynomials such that (22) holds for some $d_1, d_2 \geq 1$. Then there exists a Laurent polynomial N such that*

$$(26) \qquad L_1 = N \circ z^{D/d_1}, \qquad L_2 = N \circ z^{D/d_2},$$

where $D = LCM(d_1, d_2)$. $\qquad\square$

4. Proofs of Theorems A, B, and C

Proof of Theorem A. Suppose that (1) holds for some entire functions f, g and polynomials P, Q and let s be an entire function defined by equality (2). By Theorem 2.4 there exist an entire function h, a Laurent polynomial L, and rational functions U, V such that equalities (15) hold. Furthermore, since P, Q are polynomials, the functions U, V are Laurent polynomials. Therefore, setting

$$A = P, \quad B = Q, \quad L_1 = U, \quad L_2 = V$$

we obtain (20) and may apply Theorem 3.1.

Observe that without loss of generality we may assume that the Laurent polynomial W in Theorem 3.1 equals z. Indeed, if W is a polynomial, then the function $W \circ h$ is clearly entire, and we may simply replace h by $W \circ h$. On the other hand, if W is not a polynomial, then L is not a polynomial either, and it follows from

$$(27) \qquad s = L \circ h$$

that h omits the value 0. Therefore, in this case the function $W \circ h$ is also entire and we may replace h by $W \circ h$ as above.

If for the functions A, B, L_1, L_2 either conclusion 1) or conclusion 2) of Theorem 3.1 holds, then setting

$$F = E, \quad \tilde{P} = \tilde{A}, \quad \tilde{f} = \tilde{L}_1, \quad \tilde{Q} = \tilde{B}, \quad \tilde{g} = \tilde{L}_2, \quad t = h$$

we see that for P, Q, f, g accordingly either conclusion 1) or conclusion 2) of Theorem A holds.

On the other hand, if for the functions A, B, L_1, L_2 one of conclusions 3), 4), 5) holds, then L is not a polynomial. Therefore, h omits the value 0 and hence there exists an entire function w such that

$$(28) \qquad h = e^{iz} \circ w.$$

If 3) holds, then it follows from (28) taking into account the equalities

$$(29) \qquad U_n \circ e^{iz} = \cos(nz), \qquad V_n \circ e^{iz} = \sin(nz),$$

that

$$\tilde{L}_1 \circ h = \cos z S(\sin z) \circ w, \qquad \tilde{L}_2 \circ h = \sin z \circ w.$$

Therefore, setting

$$F = E, \quad \tilde{P} = \tilde{A}, \quad \tilde{f} = \tilde{L}_1 \circ e^{iz}, \quad \tilde{Q} = \tilde{B}, \quad \tilde{g} = \tilde{L}_1 \circ e^{iz}, \quad t = w$$

we conclude that conclusion 3) of Theorem A holds. Similarly, it follows from (28), (29) that if for A, B, L_1, L_2 conclusion 5) of Theorem 3.1 holds, then for P, Q, f, g conclusion 5) of Theorem A holds.

Finally, in order to prove that if for A, B, L_1, L_2 conclusion 4) of Theorem 3.1 holds, then for P, Q, f, g conclusion 4) of Theorem A holds, observe that the equality $\varepsilon^{nlm} = -1$ implies that $\varepsilon = e^{i\gamma}$, where

$$\gamma = \frac{\pi}{nlm}(2r+1)$$

for some r, $0 \leq r < nlm$, and hence

$$U_m \circ \varepsilon z \circ e^{iz} = U_m \circ e^{i(z+\gamma)} = \cos\left(\frac{(2k+1)\pi}{nl} + mz\right),$$

for some k, $0 \leq k < nl$.   □

Proof of Theorem B. It follows from Theorem 2.4 that if (2) holds for some entire functions s, f, g and rational functions P, Q, then there exist an entire function h, a Laurent polynomial L, and rational functions U, V such that equalities (15) hold. Furthermore, clearly either

$$(30) \qquad\qquad P \circ U \sim A \circ L_1, \quad Q \circ V \sim L_2 \circ z^d,$$

where A is a polynomial and L_1, L_2 are Laurent polynomials satisfying (21), or

$$(31) \qquad\qquad P \circ U \sim L_1 \circ z^{d_1}, \quad Q \circ V \sim L_2 \circ z^{d_2},$$

where L_1, L_2 are Laurent polynomials satisfying (22).

If (30) holds, then it follows from Theorem 3.2 that there exist rational functions $E, \hat{P}, \hat{Q}, \hat{U}, \hat{V}, W$ such that

$$(32) \qquad P = E \circ \hat{P}, \quad Q = E \circ \hat{Q}, \quad U = \hat{U} \circ W, \quad V = \hat{V} \circ W, \quad \hat{P} \circ \hat{U} = \hat{Q} \circ \hat{V}$$

and either

$$(33) \qquad\qquad \hat{P} \circ \hat{U} \sim z^n \circ z^r L(z^n), \quad \hat{Q} \circ \hat{V} \sim z^r L^n(z) \circ z^n,$$

where L is a Laurent polynomial, $r \geq 0$, $n \geq 1$, and $\mathrm{GCD}(r,n) = 1$, or

$$(34) \qquad\qquad \hat{P} \circ \hat{U} \sim T_n \circ U_m, \quad \hat{Q} \circ \hat{V} \sim U_m \circ z^n,$$

where T_n is the Chebyshev polynomial, $n \geq 1$, $m \geq 1$, and $\mathrm{GCD}(m,n) = 1$.

Observe that since in both cases the function $\hat{P} \circ \hat{U}$ is a Laurent polynomial which is not a polynomial, it follows from

$$L = E \circ \hat{P} \circ \hat{U} \circ W$$

and $L^{-1}\{\infty\} = \{0, \infty\}$ that E is a polynomial and $W = cz^{\pm d}$, where $c \in \mathbb{C}^*$, $d \geq 1$. Since (27) implies that h omits the value 0, replacing h by $W \circ h$ we may assume that $W = z$.

Setting now accordingly to possibilities (33), (34) either

$$F = E, \quad \tilde{P} = \hat{P}, \quad \tilde{f} = \hat{U} \circ e^z, \quad \tilde{Q} = \hat{Q}, \quad \tilde{g} = \hat{V} \circ e^z, \quad t = iw$$

or

$$F = E, \quad \tilde{P} = \hat{P}, \quad \tilde{f} = \hat{U} \circ e^{iz}, \quad \tilde{Q} = \hat{Q}, \quad \tilde{g} = \hat{V} \circ e^{iz}, \quad t = w,$$

where w is an entire function such (28) holds, we see that one of conclusions of Theorem B holds.

Similarly, if (31) holds, then it is not hard to prove using Lemma 3.1 that there exist rational functions $\hat{P}, \hat{Q}, \hat{U}, \hat{V}$, a Laurent polynomial E, and a Laurent

polynomial W of the form $W = cz^{\pm d}$, $d = \mathrm{GCD}(d_1, d_2)$, $c \in \mathbb{C}^*$ such that (32) holds and

$$\hat{P} \circ \hat{U} \sim z^{D/d_1} \circ z^{(d_1/d)z}, \quad \hat{Q} \circ \hat{V} \sim z^{D/d_2} \circ z^{(d_2/d)z}.$$

Furthermore, without loss of generality we may assume that $W = z$ and setting

$$F = E, \quad \tilde{P} = \hat{P}, \quad \tilde{f} = \hat{U} \circ e^z, \quad \tilde{Q} = \hat{Q}, \quad \tilde{g} = \hat{V} \circ e^z, \quad t = iw,$$

where w is an entire function such (28) holds, we see that conclusion 1) of Theorem B holds with $L \equiv 1$, $n = d_2/d$, and $r = d_1/d$. □

Proof of Theorem C. Suppose that P is a polynomial and c is a complex number such that

$$(35) \hspace{4cm} P \circ f = cP \circ g$$

for some entire functions f and g. Then for P, f, g, and $Q = cP$ one of conclusions of Theorem A holds. Furthermore, since $\deg P = \deg cP$ we have:

$$(36) \hspace{4cm} \deg \tilde{P} = \deg \tilde{Q}.$$

If 1) holds, then equality (36) together with the conditions $\mathrm{GCD}(r, n) = 1$, $r \geq 0$, $n \geq 1$ imply that

$$(37) \hspace{4cm} P \circ f \sim cP \circ g.$$

The same is true if 2) holds.

Further, it follows from (36) that 5) is impossible, while 4) may hold only if $m = n = 1$. Finally, if 3) holds, then necessarily $\deg S = 0$, and it is easy to see that in this case 3) reduces to 4) with $l = 2$, $m = n = 1$, and $k = 1$. Summing up we see that equality (35) implies that either $P \circ f$ is equivalent to $cP \circ g$ or

$$(38) \hspace{4cm} P = F \circ T_l \circ \alpha$$

for some polynomial F, Möbius transformation α, and $l \geq 2$.

If (37) holds, then there exist $a, b \in \mathbb{C}$ such that

$$(39) \hspace{4cm} cP = P \circ (az + b).$$

Set $\hat{P} = P \circ (z - b)$. Then

$$(40) \hspace{4cm} c\hat{P} = \hat{P} \circ az$$

and the comparision of coefficients of polynomials in both parts of (40) implies that there exists an nth root of unity ε such that $a = \varepsilon$, $c = \varepsilon^r$, and

$$(41) \hspace{4cm} \hat{P} = z^r R(z^n)$$

for some polynomial R and $r \geq 0$, $n > 1$. Therefore, (37) implies that

$$P = z^r R(z^n) \circ \alpha$$

for some Möbius transformation α. Furthermore, any polynomial of such a from is not a uniqueness polynomial for entire functions since for any nth root of unity ε distinct from 1 and any entire function f we have:

$$P \circ (\alpha^{-1} \circ f) = (\varepsilon^{n-r} P) \circ (\alpha^{-1} \circ \varepsilon f).$$

Finally, it is easy to see that any polynomial of form (38) also is not a strong uniqueness polynomial for entire functions since for example for the functions

$$\alpha^{-1} \circ \cos \left(\frac{2\pi}{l} + z \right), \quad g = \alpha^{-1} \circ \cos z$$

the equality

$$P \circ f = P \circ g$$

holds. $\square$

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